Minimal polynomials in spin representations of symmetric groups

Happy Birthday, Michel Broué!

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Eigenvalues of group elements

Let G be a finite group and F be an algebraically closed field of characteristic l.

For any element $g \in G$ and any irreducible representation (ρ, V) of G over F, let $\deg(\rho(g))$ be the degree of the minimal polynomial of $\rho(g)$.

Let $o(g)$ be the order of g in $G/Z(G)$, where $Z(G)$ is the center.

Since $\rho(g)^{o(g)}$ is a scalar matrix, we have $\deg(\rho(g)) \leq o(g)$.

If the order $|g|$ of g is coprime to l, then $\rho(g)$ is diagonalizable and $\deg(\rho(g))$ is equal to the number of distinct eigenvalues of $\rho(g)$.

If g is an l-element, then $\deg(\rho(g))$ is equal to the size of the largest Jordan block of $\rho(g)$.

The Classical Problem

The study of eigenvalues has a long history.

- Hall–Higman (1956): l-elements in l-solvable groups in characteristic l.
- Wales (1979): 3-cycles in irreducible spin representations of $\tilde{S}_n$.
- Thompson (2008): p-cycles in A_p with $l < p < 2l$.
- Tiep–Zalesskii (2022): p-power elements in Quasi simple classical groups with $(p, l) = 1$.

Tiep–Zalesskii Problem

Let (ρ, V) be an irreducible representation of G and $o(g)$ denote the order of g in $G/Z(G)$.

Problem (Tiep–Zalesskii)

Classify pairs (ρ, g) such that the minimal polynomial of $\rho(g)$ is of degree less than $o(g)$. In the first instance, under the condition that $o(g)$ is a prime power.

The problem should be given priority for groups that are close to being simple, such as almost simple groups, quasi-simple groups, etc.

Knowing the information about the eigenvalues of elements in representations has many applications in arithmetic geometry, number theory, and group theory, etc. For more details, see the survey articles [5, 6] by Zalesskii.

Double cover of symmetric groups

The double cover of the symmetric group S_n is a central extension of S_n by $\mathbb{Z}/2\mathbb{Z}$:

$$1 \to \mathbb{Z}/2\mathbb{Z} \to G \xrightarrow{\pi} S_n \to 1.$$

One of the solutions is as follows:

$$\begin{aligned}\tilde{S}_n = \langle -1, t_1, \ldots, t_{n-1} \mid\ & (-1)^2 = 1,\ t_i^2 = -1, \\ & (t_i t_{i+1})^3 = -1, \\ & (t_i t_j)^2 = -1 \text{ for } |i-j| > 1 \rangle.\end{aligned}$$

Irreducible spin representations

Here we recall the notation:

$$\begin{aligned}\mathrm{DP}_n &= \{\lambda \vdash n \mid \lambda_i > \lambda_{i+1} \text{ for all } i\},\\ \mathrm{DP}_n^+ &= \{\lambda \in \mathrm{DP}_n \mid n - l(\lambda) \text{ is even}\},\\ \mathrm{DP}_n^- &= \mathrm{DP}_n \setminus \mathrm{DP}_n^+,\\ \mathrm{O}_n &= \{\lambda \vdash n \mid \lambda_i \text{ is odd for all } i\}.\end{aligned}$$

$$\begin{aligned}\mathrm{Irr}(\tilde{S}_n) = \mathrm{Irr}(S_n) &\cup \ \{(\phi_\lambda, V_\lambda) \mid \lambda \in \mathrm{DP}_n^+\}\\ &\cup \ \{(\phi_\lambda^+, V_\lambda^+),\ (\phi_\lambda^-, V_\lambda^-) \mid \lambda \in \mathrm{DP}_n^-\}.\end{aligned}$$

Results for the Symmetric group

Recall that $\mathrm{Irr}(S_n) = \{(\chi_\lambda, V_\lambda) \mid \lambda \vdash n\}$, $\mathrm{Cl}(S_n) = \{C_\mu \mid \mu \vdash n\}$.

Table 1: Summary of Permutations and Eigenvalues by Author

Authors	**Permutations**	**Eigenvalues**
Klyachko (1974)	(n)	ζ_n
Swanson (2018)	(n)	all
Yang–Staroletov (2021)	(a^b)	all
Amrutha–Amri–V.S (2023)	all	1
V.S (2024)	$\mu_i \mid \mu_1$	all
Staroletov (2025)	all	all

Main theorem

Theorem (A. Prasad, A. Staroletov, V. S) (202(5+6))

Let $(\rho_\lambda, V_\lambda)$ be an irreducible spin representation of $\tilde{S}_n$ indexed by $\lambda \in \mathrm{DP}_n$. Let $g \in \tilde{S}_n$ has cycle type α, and set $k = \mathrm{lcm}(\alpha)$. Let $\varepsilon \in \{\pm 1\}$ be such that $g^k = \varepsilon$ and $n > 8$.

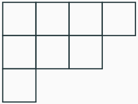
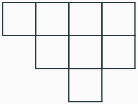
Then the minimal polynomial of $\rho(g)$ is $x^k - \varepsilon$, except in the following four cases.

1. $\lambda = (n)$, α contains 3, all other parts of α are not divisible by 3, and 5 is not the unique part divisible by 5. In this case, the minimal polynomial is $\frac{x^k - \varepsilon}{x^{k/3} - \varepsilon}$.
2. $\lambda = (n)$, α contains 5, all other parts of α are not divisible by 5, and 3 is not the unique part divisible by 3. In this case, the minimal polynomial is $\frac{x^k - \varepsilon}{x^{k/5} - \varepsilon}$.

Main theorem (contd.)

3. $\lambda = (n)$, α contains both 3 and 5, while all other parts of α are not divisible by 3 or 5. In this case, the minimal polynomial is $\frac{(x^k - \varepsilon)(x^{k/15} - \varepsilon)}{(x^{k/3} - \varepsilon)(x^{k/5} - \varepsilon)}$.
4. $\lambda = (n-1, 1)$, α contains both 3 and 5, while all other parts of α are not divisible by 3 or 5. In this case, the minimal polynomial is $\frac{x^k - \varepsilon}{x^{k/15} - \varepsilon}$.

The Analogs: Classical vs. Spin i

Symmetric Group		Double Cover
Young diagrams	$\longleftrightarrow$	Shifted Young diagrams
	$\longleftrightarrow$	
Alphabet $\{1 < 2 < \cdots < n\}$	$\longleftrightarrow$	Alphabet $\{1' < 1 < 2' < 2 < \cdots < n' < n\}$
Rows increasing, Columns strictly increasing	$\longleftrightarrow$	Rows and Columns weakly increasing, with repeated entry restrictions

The Analogs: Classical vs. Spin ii

Symmetric Group		Double Cover
Young tableaux	$\longleftrightarrow$	Shifted Young tableaux
$\begin{array}{\|c\|c\|c\|c\|}\hline 1 & 1 & 3 & 4 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array}$	$\longleftrightarrow$	$\begin{array}{cccc} \boxed{1'} & \boxed{1} & \boxed{1} & \boxed{3'} \\ & \boxed{2'} & \boxed{2} & \boxed{3'} \\ & & \boxed{3'} & \end{array}$
Murnaghan–Nakayama rule	$\longleftrightarrow$	Morris' recursion formula
$\chi_\lambda(w_n) = \begin{cases} (-1)^{n-\lambda_1} & \text{if } \lambda_2 \leq 1, \\ 0 & \text{else.} \end{cases}$	$\longleftrightarrow$	$\phi_\lambda(\sigma_n) = \begin{cases} (-1)^{n-\lambda_1} & \text{if } \ell(\lambda) \leq 2, \\ 0 & \text{else.} \end{cases}$
Littlewood–Richardson rule	$\longleftrightarrow$	Spin Littlewood–Richardson rule

Preliminary results

Let $\sigma_n = t_1 t_2 \dots t_{n-1}$.

Lemma

Order of σ_n is equal to $\begin{cases} n & \textit{if } n \equiv 0, 6, 7, 8 \pmod 8, \\ 2n & \textit{if } n \equiv 2, 3, 4, 5 \pmod 8. \end{cases}$

The following lower bound has been established by various authors.

Lemma (Lower bound)

Let $n \geq 4$. Then every irreducible spin representation has dimension at least of dimension of basic spin representation, which is $2^{\lfloor (n-1)/2 \rfloor}$.

Preliminary results (p-cycles)

Theorem

Let $p \geq 3$ be a prime. Then the minimal polynomial of σ_p in an irreducible spin representation of $\tilde{S}_p$ is $x^p \pm 1$ except when $p = 3, 5$.

Proof.

Let g be a p-cycle in $\tilde{S}_p$ of order p and ϕ be an irreducible spin representation of $\tilde{S}_p$. Then g is conjugate to g^i for every $i = 1, 2, \ldots, p-1$ and $\phi(g) \in \{0, \pm 1\}$ by Morris' recursion.

For a linear character ψ of $\langle g \rangle$, we have

$$\begin{aligned}\langle \psi, \operatorname{Res}^{\tilde{S}_p}_{\langle g \rangle} \phi \rangle &= \tfrac{\phi(1) + \phi(g)(\psi(g) + \psi(g^2) + \cdots + \psi(g^{p-1}))}{p} \\ &\geq \tfrac{\phi(1) - (p-1)}{p} \geq \tfrac{2^{(p-1)/2} - (p-1)}{p} > 0 \quad \text{for } p \geq 7.\end{aligned}$$

$\square$

The n-cycle case

Theorem (A. Prasad, A. Staroletov, V. S)

The minimal polynomial $P(x)$ of σ_n is $x^n \pm 1$ in every irreducible spin representation (ρ, V) except when

1. $\rho = \rho_{(3)}$ and $P(x) = x^2 - x + 1$.
2. $\rho = \rho^{\pm}_{(2,1)}$ and $P(x) = x + 1$.
3. $\rho = \rho^{\pm}_{(4)}$ and $P(x) = x^2 \mp \sqrt{2}x + 1$.
4. $\rho = \rho_{(5)}$ and $P(x) = \frac{x^5+1}{x+1}$.
5. $\rho = \rho^{\pm}_{(6)}$ and $P(x) = x^4 \mp \sqrt{3}x^3 - 2x^2 \pm \sqrt{3}x + 1$.
6. $\rho = \rho^{\pm}_{(8)}$ and $P(x) = \frac{x^8-1}{(x\pm 1)}$.

Moreover, for $n \geq 19$, each eigenvalue of $\rho(\sigma_n)$ has multiplicity at least n.

Spin LR-rule

For a strict partition λ, define

$$c_\lambda = \begin{cases} \sqrt{2}, & \text{if } \lambda \text{ is odd,} \\ 1, & \text{if } \lambda \text{ is even.} \end{cases}$$

Define $f^\lambda_{\mu\nu}$ to be the number of shifted tableaux of shape λ/μ and content ν such that the word of T is a 'lattice permutation'.

Theorem

Let $\mu \in \mathrm{DP}_k$, $\nu \in \mathrm{DP}_{n-k}$, and $\lambda \in \mathrm{DP}_n$. Let $\phi_\mu \times_c \phi_\nu$ be a reduced Clifford product of ϕ_μ and ϕ_ν. Then

$$\langle \phi_\mu \times_c \phi_\nu, \phi_\lambda \rangle = \frac{1}{c_\lambda c_{\mu\cup\nu}} \, 2^{(\ell(\mu)+\ell(\nu)-\ell(\lambda))/2} \, f^\lambda_{\mu\nu}, \tag{1}$$

unless λ is odd and $\lambda = \mu \cup \nu$. In that case, the multiplicity of $\phi^\pm_\lambda$ is 0 or 1 according to choice of associates.

Further questions

- **Modular case:** Generalize the main theorem to the modular case.
- **Combinatorial Description:** Is there an analog of the notion *major index* which can be used to give a combinatorial description of the eigenvalues of elements in irreducible spin representations of n-cycles?

Thank You!

Questions?

References i

P. Hall and G. Higman.
On the p-length of p-soluble groups, and a reduction theorem for burnside's problem.
Proc. London Math. Soc., 6:1–42, 1956.

A. A. Klyachko.
Lie elements in a tensor algebra.
Sibirsk. Mat. Ž., 15:1296–1304, 1430, 1974.

John G. Thompson.
Composition factors of rational finite groups.
J. Algebra, 319(2):558–594, 2008.

David B. Wales.
Some projective representations of S_n.
J. Algebra, 61(1):37–57, 1979.

A. E. Zalesski.
On eigenvalues of group elements in representations of simple algebraic groups and finite Chevalley groups.
Acta Appl. Math., 108(1):175–195, 2009.

A. E. Zalesski.
Fixed point free actions of group elements in linear groups and finite group representations.
São Paulo J. Math. Sci., 19(2):41, 2025.